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STABILITY OF MOTION OF NON-CONSERVATIVE MECHANICAL SYSTEMS†

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The stability of equilibrium of mechanical systems driven by dissipative, gyroscopic, conservative and non-conservative positional forces is investigated. Proofs are presented of several general theorems of an asymptotic nature, which state whether the systems in question are stable for sufficiently large values of the appropriate parameter. It is sometimes possible to specify bounds on the parameters that guarantee asymptotic stability or instability of equilibrium. An example is presented.

THE QUESTION of the effect of dissipative, gyroscopic and conservative forces on the stability of motion of a mechanical system is determined by the Kelvin–Chetayev theorems [1]. The presence of non-conservative positional forces considerably complicates the situation and precludes direct application of the theorems. An examination of the effect of non-conservative positional forces on the stability of equilibrium may be found, e.g. in [2–12].

1. The equations of motion of a mechnical system driven by dissipative, gyroscopic, conservative and non-conservative positional forces may be reduced to the form

$$x^{\cdot\cdot}+Bx^{\cdot}+hGx^{\cdot}+Kx+Fx=X(x,\ x^{\cdot}) \tag{1.1}$$

where $x=(x_1,\ldots,x_n)$, $B=B^T$, $G^T=-G$, $K=K^T$, $F^T=-F$ are constant matrices representing the dissipative, gyroscopic, conservative and non-conservative position forces, respectively, $X(x,x^{\cdot})$ stands for terms of at least second order in $x, x^{\cdot}$ and $h>0$ is a scalar parameter.

We will study the stability of the equilibrium state

$$x=0,\quad x^{\cdot}=0 \tag{1.2}$$

We may assume without loss of generality that $B=\operatorname{diag}(b_1,\ldots,b_n)$.

Theorem 1. If the potential energy x^TKx has a maximum at the equilibrium position, n is even and the dissipation coefficients b_i $(i=1,\ldots,n)$ are sufficiently large, then the equilibrium is unstable.

The proof is based on investigating the characteristic equation $\Delta(\lambda)=\det[E\lambda^2+(B+hG)\lambda+K+F]=0$ (E is the identity matrix). Let us calculate the coefficient of λ in the equation $\Delta(\lambda)=0$, which is $\Delta'(0)$. Using the rule for the differentiation of determinants, we obtain

$$\Delta'(0)=\sum_{i=1}^{n}\Delta_i,\quad \Delta_i=\begin{vmatrix} k_{11} & k_{12}+f_{12} & \ldots & k_{1n}+f_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ kg_{i1} & hg_{i2} & \ldots & b_i\ldots hg_{in} \\ \cdot & \cdot & \cdot & \cdot \\ k_{n1}+f_{n1} & k_{n2}+f_{n2} & \ldots & k_{nn} \end{vmatrix}$$

(k_{ij}, f_{ij}, g_{ij} are the elements of the matrices K, F and G, respectively). Expand each determinant in terms of its ith row. One of the terms in the sum will be $b_i\det(K_i+F_i)$, where K_i and F_i are the

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matrices obtained from K and F by deleting the ith row and ith column. The matrices K_i, F_i are of order $n-1$, which is odd, all the eigenvalues of K_i are negative, and F_i is skew-symmetric. The determinant of the sum of these matrices is negative, $\det(K_i+F_i)<0$ $(i=1,\ldots,n)$ [6]. Therefore, if the b_is are sufficiently large, then $\Delta'(0)<0$, which implies the existence of a root of the characteristic equation with positive real part. We note that $\Delta'(0)$ can be made negative even if only one dissipation coefficient b_i is sufficiently large, i.e. the proof of Theorem 1 remains valid in the case of partial dissipation.

Let $\det G\neq 0$, $\det F\neq 0$ and assume that the dissipation is total.

Theorem 2. If the matrices $P=Q+KG-GK$ $(Q=G^TF+F^TG)$ and $A=G^TGB+BG^TG$ are positive definite, then for sufficiently large h the equilibrium state (1.2) of system (1.1) is asymptotically stable.

To prove this we consider the following function, which is positive definite for sufficiently large h:

$$V=x^{\cdot T}(G^TG+h^{-2}K)x^{\cdot}+x^T({}^1/_2G^TG+h^{-2}K^2)x-$$
$$-2h^{-1}x^TCx^{\cdot},\quad C=FG-KG+{}^1/_2G \tag{1.3}$$

The derivative of this function with respect to time along the flow of system (1.1) is

$$\begin{aligned}V^{\cdot}=&-x^{\cdot T}[A+h^{-2}R-h^{-1}Q]x^{\cdot}-\\&-{}^1/_2h^{-1}x^TPx+2h^{-1}x^T(CB+h^{-1}FK)x^{\cdot}+\\&+2x^{\cdot T}(G^TG+h^{-2}K)X-2h^{-1}x^TCX\end{aligned} \tag{1.4}$$

$$R=KB+BK$$

which, under the assumptions of Theorem 2, is negative definite. Indeed, the quadratic part of $V^{\cdot}$ is $-x^{\cdot T}(A+h^{-2}R-h^{-1}Q)_{x^{\cdot}}-{}^1\!/\!_2h^{-1}x^TPx+2h^{-1}x^TSx^{\cdot}$. Since A and P are positive definite matrices, it follows that for sufficiently large h this quadratic form is negative definite.

Corollary. If there are no conservative forces ($K=0$), the equilibrium state (1.2) is asymptotically stable for sufficiently large h, provided that the matrices Q and A are positive definite.

For systems with two degrees of freedom ($n=2$), the positive definiteness of Q reduces to the inequality $gf>0$ (where g, f are the elements of G and F). On the other hand, the coefficient of λ in the characteristic equation is $2gf$. Hence it follows that for systems with $n=2$ the positive definiteness of Q is a necessary condition for asymptotic stability.

We will consider system (1.1) without conservative forces ($K=0$), with a positive factor $b>0$ multiplying the matrix of dissipative forces:

$$x^{\cdot\cdot}+bBx^{\cdot}+Gx^{\cdot}+Fx=X(x,x^{\cdot}) \tag{1.5}$$

Theorem 3. If the matrix $G^TB^{-1}F+F^TB^{-1}G$ is positive definite, then for sufficiently large b the equilibrium state (1.2) of system (1.5) is asymptotically stable.

Consider the function

$$V=b^{-1}(x^{\cdot}+Gx)^TB^{-1}(x^{\cdot}+Gx)+b^{-1}x^{\cdot T}x^{\cdot}+2x^Tx^{\cdot}+bx^TBx \tag{1.6}$$

The derivative $V^{\cdot}$ along the flow of system (1.5) is

$$\begin{aligned}V^{\cdot}=&-2x^{\cdot T}Bx^{\cdot}-b^{-1}x^T(G^TB^{-1}F+F^TB^{-1}G)x+2b^{-1}x^T(F+FB^{-1})x^{\cdot}+\\&+2b^{-1}(x^{\cdot}+Gx)^TB^{-1}X+2b^{-1}x^{\cdot T}X+2x^TX\end{aligned} \tag{1.7}$$

Under the assumptions of Theorem 3, the function (1.7) is negative definite. It can be shown that for sufficiently large b the function (1.6) is positive definite.

Transform variables by $x, x^{\cdot}\to u, v$: $u=x^{\cdot}+Gx$, $v=x^{\cdot}+bx$. This converts (1.6) to

$$V=b^{-1}[u^T(B+B^{-1}-E)u+v^TBv-2u^T(B-E)v+O(b^{-1})],\quad \lim_{b\to\infty}O(b^{-1})=0$$

The principal diagonal minors of the matrix of the quadratic form $u^T(B+B^{-1}-E)u+v^TBv-2u^T(B-E)v$ are: $\Delta_1=a_1,\ldots,\Delta_n=a_1a_2\ldots a_n$, $\Delta_{n+1}=a_2\ldots a_nb_1$,

$$\Delta_{n+2}=a_3\ldots a_nb_1b_2,\ \ldots\ \Delta_{2n}=b_1b_2\ldots b_n$$

$$B=\operatorname{diag}(b_1,\ldots,b_n),\quad a_i=b_i+b_{i-1}^{-1}-1$$

It follows that (1.6) is indeed positive definite for sufficiently large b.

If Q is not a positive definite matrix, the equilibrium state (1.2) of system (1.6) may be unstable.

Theorem 4. If the matrix $Q+bBF-bFB$ is negative definite, the equilibrium state (1.2) of system (1.5) is unstable.

To prove this, we consider the indefinite function

$$V=x^T(bB-G)x^{\cdot}+{}^1/_2x^T[b^2B^2-G^2-b(GB-BG)]x \tag{1.8}$$

The derivative $V^{\cdot}$ along the flow of system (1.5) is

$$V^{\cdot}=bx^{\cdot T}Bx^{\cdot}-{}^1/_2x^T(Q+bBF-bFB)x+x^T(bB-G)X$$

Under the assumptions of Theorem 4, the function (1.8) satisfies the conditions of Lyapunov's instability theorem.

We will now consider a system in which the matrix F appears together with a scalar factor $f>0$:

$$x^{\cdot\cdot}+Bx^{\cdot}+Gx^{\cdot}+Kx+fFx=X(x,\ x^{\cdot}) \tag{1.9}$$

Theorem 5. If $\det F\neq 0$, then for sufficiently large f the equilibrium state (1.2) of system (1.9) is unstable, regardless of the dissipative, gyroscopic and conservative forces.

Consider the indefinite function

$$V=x^T(E+F)x^{\cdot}+{}^1/_2x^TBx$$

The derivative $V^{\cdot}$ along the flow of system (1.9) is

$$V^{\cdot}=x^{\cdot T}x^{\cdot}+fx^TF^TFx-x^T(G+FB+FG)x^{\cdot}-x^T(K+FK)x+x^T(E+F)X \tag{1.10}$$

which, under the assumptions of Theorem 5, is positive definite.

Let us assume that the only forces acting on our mechanical system are conservative forces and non-conservative positional forces. The equations of the perturbed motion in normal coordinates, to a first approximation are

$$x^{\cdot\cdot}+Kx+Fx=0,\quad K=\operatorname{diag}(\lambda_1,\ldots,\lambda_n) \tag{1.11}$$

Suppose that $\lambda_i>0$ $(i=1,\ldots,n)$ and that no two of these numbers are equal. We may assume without loss of generality that $\lambda_1>\lambda_2>\ldots>\lambda_n$. The present author [10] established the following condition for the equilibrium state (1.2) of system (1.11) to be stable:

$$\|F\|<{}^1/_2\{(\lambda_1+\lambda_n)^2+2\lambda_ns\}^{1/2}-{}^1/_2(\lambda_1+\lambda_n) \tag{1.12}$$

$$\|F\|=\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|f_{ij}|,\quad s=\min_{i\neq j}|\lambda_i-\lambda_j|$$

Theorem 6. If s is sufficiently large, the equilibrium of system (1.11) is stable.

This follows from inequality (1.12).

Let $\lambda_n=0$, but $\lambda_1>\lambda_2>\ldots>\lambda_{n-1}>0$. Define the norm of the matrix F by its last row: $\|F\|=|f_{n1}|+\ldots+|f_{nn-1}|$.

Theorem 7. If $\|F\|$ is sufficiently large, the equilibrium of system (1.11) is stable.

This follows from the stability condition for the equilibrium of system (1.11) in the case $\lambda_n=0$ (inequality (2.10) in [10]). The same inequality also implies an upper bound for $\|F\|$, below which the equilibrium state (1.2) will be stable.

2. The statements of Theorems 2, 3 and 5 and the corollary to Theorem 2 are asymptotic in nature. It is clearly important to find lower bounds for the scalar parameters h, b and f in Eqs (1.1), (1.5) and (1.9), above which the equilibrium state (1.2) are asymptotically stable or unstable. To that end we will use the functions V constructed in the proofs of the theorems. As we have throughout adopted a unified approach, it will suffice to demonstrate the derivation of such bounds for h and f only (Theorems 2 and 5).

We first return to Theorem 2 and determine a lower bound for h, above which the equilibrium state (1.2) of system (1.1) is asymptotically stable. Let g_0, k_1, k_0 be the least eigenvalues of G^TG and K, respectively, and the eigenvalue of K of least absolute value. The function (1.3) satisfies the inequalities

$$\begin{aligned}2V &\geqslant 2x^{\cdot T}(G^TG+h^{-2}K)x^{\cdot}+ax^Tx-4h^{-1}x^TCx^{\cdot}=2x^{\cdot T}[G^TG+h^{-2}(K-2a^{-1}C^TC)]x^{\cdot}+\\ &+y^Ty\geqslant 2[g_0+h^{-2}(k_1-2a^{-1}c_1)]x^{\cdot T}x^{\cdot}+y^Ty\end{aligned} \tag{2.1}$$

We have used the notation

$$y=a^{1/2}x-2h^{-1}a^{-1/2}Cx^{\cdot},\quad a=g_0+2h^{-2}k_0^2$$

and denoted the largest eigenvalue of C^TC by c_1. The function V is positive definite provided that

$$\varphi_1(h)=g_0^2h^4+(2k_0^2g_0+g_0k_1-2c_1)h^2+2k_1k_0^2>0 \tag{2.2}$$

If this inequality holds for any $h>0$, the function (1.3) will be positive definite. Otherwise it will be positive definite for $h>h_1$, where h_1 is the largest positive root of the equation $\varphi_1(h)=0$. The quadratic part of $V^{\bullet}$, which has the form of (1.4), satisfies the inequalities

$$\begin{aligned}-V_2^{\bullet}&\geqslant {}^1/_2h^{-1}\mu_0x^Tx-h^{-1}x^TC_0x^{\cdot}+x^{\cdot T}(A-h^{-1}Q+h^{-2}R)x^{\cdot}=\\ &={}^1/_2h^{-1}(\mu_0^{1/2}x-\mu_0^{-1/2}C_0x^{\cdot})^T(\mu_0^{1/2}x-\mu_0^{-1/2}C_0x^{\cdot})+x^{\cdot T}[A-h^{-1}Q+h^{-2}R-\\ &-{}^1/_2\mu_0^{-1}h^{-1}C_1^TC_1-\mu_0^{-1}h^{-2}(C_1^TFK-KFC_1)-2\mu_0^{-1}h^{-3}KF^TFK]x^{\cdot}\geqslant\\ &\geqslant {}^1/_2h^{-1}(\mu_0^{1/2}x-\mu_0^{-1/2}C_0x^{\cdot})^T(\mu_0^{1/2}x-\mu_0^{-1/2}C_0x^{\cdot})+\\ &+[b_0-h^{-1}(\mu_1+{}^1/_2\mu_0^{-1}c_2)+h^{-2}k_2-2\mu_0^{-1}h^{-3}f_1]x^{\cdot T}x^{\cdot},\end{aligned} \tag{2.3}$$

$$C_0=C_1+2h^{-1}FK,\quad C_1=GB+2FGB-2KGB$$

where μ_0, b_0, k_2 are the least eigenvalues of the matrices P, A, $R+\mu_0^{-1}(KFC_1-C_1^TFK)$, respectively; μ_1, c_2 and f_1 are the largest eigenvalues of the matrices Q, $C_1^TC_1$, KF^TFK.

It follows from inequalities (2.3) that the function (1.4) will be negative definite provided that

$$\varphi_2(h)=2b_0\mu_0h^3-(c_2+2\mu_0\mu_1)h^2+2\mu_0k_2h-4f_1>0$$

Define $h_0=\max(h_1, h_2)$, where h_2 is the largest positive root of the equation $\varphi_2(h)=0$. Then the equilibrium state (1.2) of system (1.1) is asymptotically stable for $h>h_0$.

If $K=0$, then $h_1=(2c_1)^{1/2}g_0^{-1}$, $h_2=(c_2+2\mu_0\mu_1)/(2b_0\mu_0)$.

The derivative $V^{\bullet}$ defined by (1.10) satisfies the inequality

$$\begin{aligned}V^{\bullet}&=(x^{\cdot}-{}^1/_2C_2^Tx)^T(x^{\cdot}-{}^1/_2C_2x)+x^T[fF^TF-K-{}^1/_2FK+{}^1/_2KF-{}^1/_4C_2C_2^T]x+\\ &+x^T(E+F)X\geqslant(x^{\cdot}-{}^1/_2C_2^Tx)^T(x^{\cdot}-{}^1/_2C_2^Tx)+\\ &+(f_0f-d_1-{}^1/_4d_0)x^Tx+x^T(E+F)X,\quad C_2=G+FB+FG\end{aligned} \tag{2.4}$$

where f_0 is the least eigenvalue of F^TF, d_0 and d_1 are the largest eigenvalues of $C_2C_2^T$ and $K+\frac{1}{2}(FK-KF)$, respectively.

The follows from inequality (2.4) that $V^{\bullet}$ is positive definite for $f>(d_0+4d_1)/(4f_0)$.

3. *Example.* Let us investigate the stability of a power-driven gyroscopic horizon. A description of this mechanical system may be found in [13], where the precessional equations of motion of a system mounted on a mobile base were studied. The full equations of perturbed motion of the system on a stationary base may be reduced to the form

$$x^{\cdot\cdot}+Bx^{\cdot}+hGx^{\cdot}+Fx=X(x,\ x^{\cdot})$$

$$x=(\alpha,\ \beta,\ \gamma,\ \delta)^{\tau} \tag{3.1}$$

where α and β are the angles between the platform and the level plane, γ and δ are the angles of precession of each pair of gyroscopes, which are coupled together by antiparallelograms, h is the angular momentum of the gyroscope and B is the matrix of dissipative forces obtained when allowance is made for friction in the suspension axes of the platform and the gyroscopes. The platform and the gyros are controlled with the help of motors, producing non-conservative positional forces, the regulation being such that the matrix Q is positive definite. By the Corollary to Theorem 2 the equilibrium position (1.2) of system (3.1) is asymptotically stable for sufficiently large h.

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